

Design and Implementation of Repetitive Control based Noncausal Zero-Phase Iterative Learning Control

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Abstract—Discrete-time domain Iterative Learning Control (ILC) schemes inspired by Repetitive control algorithms are proposed and analyzed. The well known relation between a discrete-time plant (filter) and its Markov Toeplitz matrix representation has been exploited in previous ILC literature. However, this connection breaks down when the filters have noncausal components. In this paper we provide a formal representation and analysis that recover the connections between noncausal filters and Toeplitz matrices. This tool is then applied to translate the anti-causal zero-phase-error prototype repetitive control scheme to the design of a stable and fast converging ILC algorithm. The learning gain matrices are chosen such that the resulting state transition matrix has a *Symmetric Banded Toeplitz* (SBT) structure. It is shown that the well known sufficient condition for repetitive control closed loop stability based on a filter's frequency domain H_∞ norm is also sufficient for ILC convergence and that the condition becomes necessary as the data length approaches infinity. Thus the ILC learning matrix design can be translated to repetitive control loop shaping filter design in the frequency domain.

NOMENCLATURE

In this paper upper case symbols (e.g., G) represent discrete transfer functions. Upper case bold symbols (e.g., \mathbf{G}) represent corresponding matrices and lower case bold symbols (e.g., \mathbf{u}) represent vectors in the “lifted” domain. Lower case symbols (e.g., b) represent scalar values. Also $\mathbf{0}$ and \mathbf{I} stand for the zero and identity matrices respectively. In general $\mathbf{M}_{a,b}$ stands for a matrix of dimensions $a \times b$.

I. INTRODUCTION

Iterative Learning Control is a common methodology used in reducing tracking errors trial-by-trial for systems that operate repetitively [1], [2], [3]. In such systems, the reference is usually unchanged from one iteration to the next. An expository view of common ILC schemes is provided in [4], [5]. General convergence conditions based on operator norm for different ILC schemes can be found in [6]. The relationship between learning filter design in the “lifted” domain and causal filter

design in the frequency domain is detailed in [7], [8], [9], [10]. We wish to extend this relationship to noncausal filter design in the time axis. Higher order ILC schemes both in time and iteration axis are described in [11], [12], [13]. The learning filter design in [13] is based on minimization of the tracking error norm. For monotonic convergence, high-order ILC schemes in iteration axis alone may not suffice [14], [15].

Causal ILC laws result in the transition matrix having a lower triangular Toeplitz structure. The Toeplitz matrix structure and its commutability is lost when we deal with noncausal filters. In this paper, a treatment for this difficulty is proposed and also used to facilitate the ILC design. This design is analogous to noncausal zero-phase error compensation and filtering used in repetitive control design [16]. Connections between repetitive control and ILC are well established in the literature [17], [18], [19], [20], [21], [22]. Zero-phase based ILC laws are known to result in good transient behavior and have nice robustness properties. But thus far they have been developed for infinite time signals that can only be applied to sufficiently long finite time signals [23]. Using noncausal operators in ILC design is in itself not a new idea [24], [25]. As reported in [25], [26], there is good reason to consider noncausal operators especially for non-minimum phase systems and also for minimum phase systems with high relative degree. In particular, zero-phase noncausal filters have been employed as learning gains for servo-positioning applications [27], [28], [29]. But most of the earlier ILC work on noncausal filters make use of infinite-time domain (continuous or discrete) theory and employ related frequency domain conditions to arrive at conditions for iteration convergence. It is usually (tacitly) assumed that the same conditions will hold when the learning controller is implemented using discrete finite-time signals. In this context, we point out some crucial observations made in [17]:

- 1) Stability condition for ILC based on steady state frequency response strictly applies only to infinite-time signals.
- 2) Conclusions based on frequency domain conditions apply only to parts of the trajectory for which steady state frequency response describes the input-output relation.

For causal filters, the exact relationship between ILC convergence (true stability condition) and the frequency domain (approximate stability condition) are brought out in the same paper. Similar arguments can be made for noncausal ILC as well and hence one cannot assume, in general, that the approx-

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imate stability condition is a sufficient condition for iteration error convergence. We therefore propose a special choice of the learning gain matrices that translates the ILC matrix design to repetitive control loop shaping design in the frequency domain. In particular, the noncausal learning gain matrices we choose result in a symmetric banded toeplitz (SBT) cycle-to-cycle transition matrix. This special structure enables us to use the frequency domain approximate stability condition as a sufficient condition for iteration convergence. Furthermore, we show that the approximate stability condition becomes both necessary and sufficient for iteration convergence when the data length approaches infinity. The scheme presented herein was successfully implemented on a dual stage fast tool servo system [30].

The rest of this paper is organized as follows: Section II introduces common ILC based on causal compensations such as P or PD type ILC. Section III reviews the so called prototype repetitive control and motivates an analogous ILC scheme. Section IV reviews the repetitive control algorithm modified for robustness and the central theme, the modified repetitive ILC algorithm. Section V provides a convergence analysis of the proposed algorithm with sufficient conditions established in the frequency domain. Section VI establishes conditions for monotonic convergence of the error for a special choice of learning gain matrix. Section VII gives a simple simulation example to bring out a key requirement for the said convergence. Section VIII details the experimental verification of the proposed scheme and a simulation which shows how the ILC scheme can improve over stable inversion based feedforward control.

II. PROBLEM SETUP

Let us define the learning control, plant output and desired output by supervectors \mathbf{u}_k , \mathbf{y}_k and \mathbf{r} respectively.

$$\begin{aligned}\mathbf{u}_k &= [u(0) \quad \dots \quad u(n-1)]^T \\ \mathbf{r} &= [r(d) \quad \dots \quad r(n+d-1)]^T \\ \mathbf{y}_k &= [y(d) \quad \dots \quad y(n+d-1)]^T\end{aligned}\quad (1)$$

where n is the length of each trial, k is the iteration index and d is the relative degree of the plant. The plant output can then be represented in the so called “lifted” domain (introduced in [31] as a mathematical framework to represent ILC systems) as

$$\mathbf{y}_k = \mathbf{G}\mathbf{u}_k \quad (2)$$

Where \mathbf{G} is a lower triangular matrix of Markov parameters of the Linear Time Invariant (LTI) plant given by

$$\begin{bmatrix} h_d & 0 & 0 & \dots & 0 \\ h_{d+1} & h_d & 0 & \dots & 0 \\ h_{d+2} & h_{d+1} & h_d & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{n+d-1} & h_{n+d-2} & h_{n+d-3} & \dots & h_d \end{bmatrix} \quad (3)$$

We assume the initial condition is the same for each iteration and hence it does not appear in the error propagation. We shall

ignore it hereafter as is commonly done in the literature. Now a generic ILC update law would be of the form

$$\mathbf{u}_{k+1} = \mathbf{T}_u \mathbf{u}_k + \mathbf{T}_e (\mathbf{r} - \mathbf{y}_k) \quad (4)$$

where \mathbf{T}_u and \mathbf{T}_e are square matrices. This leads to the propagation equation

$$\begin{aligned}\mathbf{u}_{k+1} &= (\mathbf{T}_u - \mathbf{T}_e \mathbf{G}) \mathbf{u}_k + \mathbf{T}_e \mathbf{r} \\ \mathbf{e}_{k+1} &= \mathbf{r} - \mathbf{G}(\mathbf{T}_u \mathbf{u}_k + \mathbf{T}_e \mathbf{e}_k) \\ &= (\mathbf{T}_u - \mathbf{G} \mathbf{T}_e) \mathbf{e}_k + (\mathbf{I} - \mathbf{T}_u) \mathbf{r} \\ &\quad \text{if } \mathbf{G}, \mathbf{T}_u \text{ commute}\end{aligned}\quad (5)$$

We note that two matrices commute if they are lower triangular Toeplitz or circulant [32]

A. Arimoto P-type ILC Algorithm

First we define the tracking error as the vector

$$\mathbf{e}_k = \mathbf{r} - \mathbf{y}_k \quad (6)$$

The Arimoto P-type learning law [1] is given by

$$u_{k+1}(t) = u_k(t) + \alpha e_k(t) \quad (7)$$

which we get by substituting $\mathbf{T}_u = \mathbf{I}$ and $\mathbf{T}_e = \alpha \mathbf{I}$ in (4). The tracking error propagates according to

$$\begin{aligned}\mathbf{e}_{k+1} &= \mathbf{r} - \mathbf{y}_{k+1} \\ &= \mathbf{r} - \mathbf{G} \mathbf{u}_{k+1} \\ &= \mathbf{r} - \mathbf{G} \mathbf{u}_k - \alpha \mathbf{G} \mathbf{e}_k \\ &= (\mathbf{I} - \alpha \mathbf{G}) \mathbf{e}_k\end{aligned}\quad (8)$$

A necessary and sufficient condition for iteration convergence is that the state transition matrix $(\mathbf{I} - \alpha \mathbf{G})$ be stable. Since the state transition matrix is lower triangular this translates to $|1 - \alpha h_d| < 1$. If it is stable, we have the fixed point of the iteration given by

$$\begin{aligned}\mathbf{e}_\infty &= 0 \\ \mathbf{u}_\infty &= \mathbf{u}_\infty + \alpha(\mathbf{r} - \mathbf{G} \mathbf{u}_\infty) \\ &= \mathbf{G}^{-1} \mathbf{r}\end{aligned}\quad (9)$$

The convergence condition can be easily met by an appropriate choice of the learning gain α . Although this is a necessary and sufficient for iteration convergence, the convergence is not necessarily monotonic. To guarantee monotonicity [33] we require

$$|1 - \alpha h_d| + |\alpha| \sum_{i=1}^{n-1} |h_{d+i}| < 1 \quad (10)$$

If the Markov parameter h_d is close to zero, the matrix \mathbf{G}^{-1} becomes numerically unstable (very large entries) leading to undesirable control input \mathbf{u}_∞ . This could lead to actuator saturation and hence may not be implementable in practise. We could instead have a PD-type learning law [13]

$$u_{k+1}(t) = u_k(t) + \alpha e_k(t) + \beta e_k(t-1)$$

which would result in a 2-band \mathbf{T}_e matrix with zeros above the main diagonal. We can extend the update law to being noncausal in time which translates to \mathbf{T}_e and possibly \mathbf{T}_u matrices having non-zero elements above the main diagonal. For example, the so called zero-phase filter of order nq

$$\begin{aligned} Q(z, z^{-1}) &= q_0 + q_1(z + z^{-1}) + \dots + q_{nq}(z^{nq} + z^{-nq}) \\ \sum_{i=0}^{nq} q_i &= 1 \end{aligned} \quad (11)$$

will result in a symmetric banded Toeplitz matrix. There exists a sufficient condition for iteration convergence in the frequency domain when using causal filters [8]. We would like to extend this result to the noncausal update law as well. It turns out that this transition is not straightforward and the reasons are elucidated later. If we were to restrict the noncausal filters to be of the form (11) the resulting state transition matrix has a favorable structure which enables us to derive sufficient conditions for iteration convergence.

III. PROTOTYPE REPETITIVE CONTROL

Repetitive control [34], [35] is a special type of controller that handles periodic signals based on the internal model principle [36]. The repetitive control scheme rejects disturbances appearing at a known fundamental frequency and its harmonics. Given the stable causal plant transfer function

$$G(z^{-1}) = z^{-d}G^+(z^{-1})G^-(z^{-1}) \quad (12)$$

where d is the relative degree of the plant and we have split the transfer function into invertible and non-invertible parts. The prototype repetitive controller [35] based on stable plant inversion is given by

$$\begin{aligned} K_{rep}(z^{-1}) &= \alpha \frac{z^{-N+d}G^-(z)}{bG^+(z^{-1})(1-z^{-N})} \\ b &= \max_{\omega} |G^-(e^{-j\omega})|^2, \quad \omega \in [0, \pi] \end{aligned} \quad (13)$$

This can be written in feedback form as

$$u(t) = u(t-N) + \alpha \frac{[z^{-nu}G^-(z)]}{bG^+(z^{-1})} e(t-N+d+nu) \quad (14)$$

where d is the relative degree of the plant, nu denotes the number of unstable zeros and N is the period. Introducing the variable $u'(t) = bG^+(z^{-1})u(t)$ we get

$$u'(t) = u'(t-N) + \alpha [z^{-nu}G^-(z)] e(t-N+d+nu) \quad (15)$$

Note the abuse of notation in (14) and (15) where product of a transfer function and a scalar function of time is intended to convey convolution.

A. Prototype Iterative Learning Control Algorithm

Analogous to (12) we have in the “lifted” domain

$$\mathbf{G} = \mathbf{G}^+\mathbf{G}^- \quad (16)$$

Note that the invertible part of the system $G^+(z^{-1})$ is represented by the lower triangular matrix \mathbf{G}^+ consisting of its

Markov parameters (3). The all zero part of the plant $G^-(z^{-1})$ is represented by the lower triangular banded Toeplitz matrix

$$\begin{aligned} \mathbf{G}_{ij}^- &= g_{i-j}, \quad 0 \leq i-j \leq nu \\ &= 0, \quad \text{otherwise} \end{aligned} \quad (17)$$

Hence the plant output defined in (2)

$$\mathbf{y}_k = \mathbf{G}\mathbf{u}_k = \mathbf{G}^-\mathbf{G}^+\mathbf{u}_k = \mathbf{G}^-\mathbf{u}'_k$$

where we have introduced the variable $\mathbf{u}' = \mathbf{G}^+\mathbf{u}$. Analogous to (15) we propose the prototype learning law

$$\mathbf{u}'_{k+1} = \mathbf{u}'_k + \alpha(\mathbf{G}^-)^T \mathbf{e}_k \quad (18)$$

We will address the stability of the prototype ILC in section V. Assuming it is indeed stable the fixed point of iterations is given by

$$\begin{aligned} (\mathbf{G}^-)^T \mathbf{e}_\infty &= 0 \\ (\mathbf{G}^-)^T \mathbf{G}^- \mathbf{u}'_\infty &= (\mathbf{G}^-)^T \mathbf{r} \end{aligned} \quad (19)$$

Hence we see that the prototype ILC, if stable, will converge to the optimal solution of the least squares minimization problem

$$\min_{\mathbf{u}'} \|\mathbf{r} - \mathbf{G}^-\mathbf{u}'\|_2 \quad (20)$$

IV. MODIFIED REPETITIVE CONTROL

To incorporate robustness in the presence of plant model uncertainty, the prototype repetitive control (15) was modified as follows [34]

$$\begin{aligned} u'(t) &= [z^{-nq}Q(z, z^{-1})] \{u'(t-N+nq) \\ &+ \alpha [z^{-nu}G^-(z)] e(t-N+d+nu+nq)\} \end{aligned} \quad (21)$$

where $Q(z, z^{-1})$ is a zero-phase low pass filter(11). The corresponding matrix in the “lifted” domain is

$$\begin{aligned} \mathbf{Q}_{ij} &= q_{|i-j|}, \quad |i-j| \leq nq \\ &= 0, \quad \text{otherwise} \end{aligned} \quad (22)$$

We wish to establish a learning control law inspired by (21) which would result in a easy to check stability condition. In the next section we introduce the central focus of this paper viz., the modified repetitive ILC algorithm. We have taken great care in setting up the proposed algorithm so as to result in an elegant and easy to check condition for iteration convergence.

A. Modified Repetitive ILC Algorithm

First we define learning gain filters Q_u and Q_e , of the form (11), of orders nq^u and nq^e respectively. We redefine the extended plant input, output and reference signals by

$$\begin{aligned} \mathbf{u}_k &= [u(0) \quad \dots \quad u(n+2nu-1)]^T \\ \mathbf{y}_k &= [y(1) \quad \dots \quad y(n+2nu)]^T \\ \mathbf{r} &= [r(1) \quad \dots \quad r(n+2nu)]^T \end{aligned}$$

where we have assumed the relative degree $d = 1$ without any loss of generality. Now we zero-pad the control input as follows

$$\begin{aligned} \mathbf{u}'_k(t) &= 0, & t &= 0 \dots nu-1 \\ &= \bar{\mathbf{u}}_k(t-nu), & t &= nu \dots n+nu-1 \\ &= 0, & t &= n+nu \dots n+2nu-1 \end{aligned}$$

which gives us the modified plant model

$$\mathbf{y}_k = \mathbf{G}^- \mathbf{u}'_k = \mathbf{G}^- \mathbf{N} \bar{\mathbf{u}}_k$$

where the “zero-padding” matrix \mathbf{N} is defined according to

$$\mathbf{N}_{n+2nu,n} = \begin{bmatrix} \mathbf{0}_{nu,n} \\ \mathbf{I}_{n,n} \\ \mathbf{0}_{nu,n} \end{bmatrix}$$

Now we propose the learning control law

$$\bar{\mathbf{u}}_{k+1} = \mathbf{Q}_u \bar{\mathbf{u}}_k + \mathbf{F} \mathbf{e}_k \quad (23)$$

where \mathbf{Q}_u is the $n \times n$ matrix representation of Q_u as in (22). We can not derive an error propagation relation because $\mathbf{G}^- \mathbf{N}$ and \mathbf{Q}_u do not commute in general. Instead we shall use a state space approach to establish stability where $\bar{\mathbf{u}}$, \mathbf{r} and \mathbf{e} will serve as the system state, input and output respectively. From (6) and (23) we have the control propagation equation

$$\begin{aligned} \bar{\mathbf{u}}_{k+1} &= \mathbf{Q}_u \bar{\mathbf{u}}_k + \mathbf{F} \mathbf{e}_k \\ &= \mathbf{Q}_u \bar{\mathbf{u}}_k + \mathbf{F}(\mathbf{r} - \mathbf{y}_k) \\ &= \mathbf{Q}_u \bar{\mathbf{u}}_k - \mathbf{F} \mathbf{G}^- \mathbf{u}'_k + \mathbf{F} \mathbf{r} \\ &= (\mathbf{Q}_u - \mathbf{F} \mathbf{G}^- \mathbf{N}) \bar{\mathbf{u}}_k + \mathbf{F} \mathbf{r} \end{aligned} \quad (24)$$

Now we choose $\mathbf{F} = \alpha \mathbf{N}^T (\mathbf{G}^-)^T \mathbf{Q}_e$ where \mathbf{Q}_e is the $(n + 2nu) \times (n + 2nu)$ matrix representation of Q_e . we end up with the state transition matrix $\mathbf{A} = \mathbf{Q}_u - \alpha \mathbf{N}^T (\mathbf{G}^-)^T \mathbf{Q}_e \mathbf{G}^- \mathbf{N}$. As was observed in [6] if $\mathbf{Q}_u \neq \mathbf{I}_{n,n}$ the error vector no longer converges to the zero vector. A necessary and sufficient condition for iteration convergence is that the state transition matrix \mathbf{A} be stable. It is worth noting that the “zero-padding” matrix \mathbf{N} and the learning gains \mathbf{Q}_u and \mathbf{Q}_e have all been chosen carefully such that \mathbf{A} has a SBT structure. It is this special structure that establishes the vital link between the “lifted” and frequency domain stability conditions.

V. STABILITY ANALYSIS

A. Symmetric Banded Toeplitz Structure

It can be shown that the state transition matrix \mathbf{A} has the SBT structure below (c.f. Appendix A)

$$\begin{bmatrix} a_0 & a_1 & \dots & a_r & & & \\ a_1 & a_0 & & & & & \\ \vdots & & \ddots & & \ddots & & \\ a_r & & & & & & 0 \\ & \ddots & & & & & \\ & & a_r & \dots & a_1 & a_0 & a_1 & \dots & a_r \\ & & & & 0 & & & \ddots & \\ & & & & & & & & a_r \\ & & & & & & & & \vdots \\ & & & & & & & & a_0 & a_1 \\ & & & & & & & & a_r & \dots & a_1 & a_0 \end{bmatrix}$$

where $r = \max(nu, nq^e + nu)$. The entries of the matrix above, a_i , have been explicitly derived in Appendix B.

B. Connections to Frequency Domain Stability Condition

A necessary and sufficient condition for iteration convergence is

$$\max_m |\lambda_m(\mathbf{A})| < 1, \quad m = 1 \dots n \quad (25)$$

where λ_m stands for the m^{th} eigenvalue of \mathbf{A} . This is the true stability condition (using the terminology in [17]). Now let the discrete time noncausal filter representation of \mathbf{A} be

$$A(z, z^{-1}) = a_0 + \sum_{k=1}^r a_k (z^k + z^{-k})$$

where the discrete time transform variable is evaluated at $z = e^{-j\omega T}$. Now for sufficiency we look at the H_∞ norm condition [7], [8]

$$|A(e^{-j\omega T})| < 1 \quad \forall \quad \omega T \in [0, \pi]$$

which translates to

$$|a_0 + 2 \sum_{k=1}^r a_k \cos(k\theta)| < 1 \quad \forall \quad \theta \in [0, \pi] \quad (26)$$

We use a unique property of symmetric Toeplitz matrices (c.f. Section 4.2, Lemma 4.1 [32]) viz.,

$$\max_m |\lambda_m(\mathbf{A})| < \max_{\theta \in [0, \pi]} |a_0 + 2 \sum_{k=1}^r a_k \cos(k\theta)| \quad (27)$$

which gives us the required *key* link between frequency domain and the “lifted” domain. For the choice of learning matrices we made, the above translates to the easy to check H_∞ norm condition

$$|Q_u(z, z^{-1}) - \alpha Q_e(z, z^{-1}) G^-(z^{-1}) G^-(z)|_\infty < 1 \quad (28)$$

Using the terminology in [17], the above would be the approximate stability condition. For the special choice $Q_u = Q_e = 1$ we get back the prototype ILC (c.f. section III-A) with the simple stability condition

$$|1 - \alpha G^-(z^{-1}) G^-(z)|_\infty < 1 \quad (29)$$

which holds iff $\alpha \in [0, 2]$ [35]. We immediately note that for a general system it is easier to satisfy (28) because of the extra degrees of freedom in choosing Q_u and Q_e .

The above result is only a sufficient condition for error convergence. The main obstacle in proving necessity is the absence of explicit formulae for the eigenvalues of the matrix \mathbf{A} . To show that the condition is not overly conservative we shall first perturb \mathbf{A} to convert it into a circulant matrix [32] and second look at the special case when \mathbf{A} is a tri-diagonal matrix. In both situations we have the luxury of being able to compute the eigenvalues explicitly and show that the condition is fairly stringent. Not surprisingly, efficient computation of the eigenvalues of a SBT matrix is in itself a well researched topic [37], [38].

C. Circulant Matrix Approximation to the Transition Matrix

We can approximate \mathbf{A} by embedding it in the circulant matrix $\tilde{\mathbf{A}}$ below [38]

$$\begin{bmatrix} a_0 & a_1 & \dots & a_r & & & a_r & \dots & a_1 \\ a_1 & a_0 & & & & & & \ddots & \vdots \\ \vdots & & \ddots & & \ddots & & 0 & & a_r \\ a_r & & & & & & & & \\ & \ddots & & & & & & & \\ & & a_r & \dots & a_1 & a_0 & a_1 & \dots & a_r \\ & & & 0 & & & \ddots & & a_r \\ a_r & & & & & & & & \vdots \\ \vdots & \ddots & & & & & & a_0 & a_1 \\ a_1 & \dots & a_r & & & & a_r & \dots & a_1 & a_0 \end{bmatrix}$$

We note this is not a bad approximation considering that in practice $n \gg r$ and the added terms do not perturb the eigenvalues too much. In fact we have for $n \gg r$ [32]

$$\max_m |\lambda_m(\mathbf{A})| \simeq \max_m |\lambda_m(\tilde{\mathbf{A}})|$$

The eigenvalues of $\tilde{\mathbf{A}}$ are given by the Discrete Fourier Transform (DFT) of the first row of the matrix [9], [32]

$$\begin{aligned} \lambda_m(\tilde{\mathbf{A}}) &= a_0 + \sum_{k=1}^r a_k \left(e^{-\frac{2\pi i k(m-1)}{n}} + e^{-\frac{2\pi i (n-k)(m-1)}{n}} \right) \\ &= a_0 + 2 \sum_{k=1}^r a_k \cos\left(\frac{2\pi k(m-1)}{n}\right), \quad m = 1 \dots n \end{aligned}$$

Hence the condition for convergence becomes

$$\max_m |a_0 + 2 \sum_{k=1}^r a_k \cos(k\theta_m)| < 1, \quad \theta_m = \frac{2\pi(m-1)}{n}$$

When $\theta_m \in [0, \pi]$ we readily have the sufficient condition for iteration convergence from (26). When $\theta_m \in [\pi, 2\pi]$ let $\theta_m = 2\pi - \theta$ where $\theta \in [0, \pi]$. Then we have

$$\cos(k\theta_m) = \cos(k2\pi - k\theta) = \cos(k\theta)$$

which gives us the required sufficient condition for iteration convergence. In the limiting case when $n \rightarrow \infty$, we see that θ_m spans the entire range $(0, \pi)$ and hence the norm condition (26) becomes both necessary and sufficient.

D. Three Banded Transition Matrix

If the system were such that $r = 1$ then the resulting state transition matrix is tridiagonal with eigenvalues given by [39]

$$\lambda_m(\mathbf{A}) = a_0 + 2a_1 \cos(\theta_m), \quad \theta_m = \left(\frac{m\pi}{n+1} \right), \quad m = 1 \dots n$$

From (26) we have the H_∞ norm condition

$$|a_0 + 2a_1 \cos(\theta)| < 1 \quad \forall \quad \theta \in [0, \pi] \quad (30)$$

which gives a sufficient condition for iteration convergence. In the limiting case when $n \rightarrow \infty$, we again see that θ_m spans the entire range $(0, \pi)$ and hence the norm condition becomes both necessary and sufficient.

VI. MONOTONIC ERROR CONVERGENCE

For the special choice $\mathbf{Q}_u = \mathbf{I}_{n,n}$ the modified Repetitive ILC algorithm (see section IV-A) simplifies to

$$\begin{aligned} \bar{\mathbf{u}}_{k+1} &= \bar{\mathbf{u}}_k + \mathbf{F}\mathbf{e}_k \\ &= (\mathbf{I} - \mathbf{F}\mathbf{G}^{-1}\mathbf{N})\bar{\mathbf{u}}_k + \mathbf{F}\mathbf{r} \\ &= \mathbf{A}\bar{\mathbf{u}}_k + \mathbf{F}\mathbf{r} \end{aligned} \quad (31)$$

Now if assume the initial conditions $\bar{\mathbf{u}}_0 = \mathbf{0}$ and $\mathbf{e}_0 = \mathbf{r}$ we can show by induction that

$$\mathbf{F}\mathbf{e}_k = \mathbf{A}^k \mathbf{F}\mathbf{r} \quad (32)$$

which gives us the error propagation equation

$$\mathbf{F}\mathbf{e}_{k+1} = \mathbf{A}\mathbf{F}\mathbf{e}_k \quad (33)$$

Since \mathbf{A} is symmetric by definition we have,

$$\begin{aligned} \|\mathbf{A}\|_2^2 &= \lambda_{\max}(\mathbf{A}^T \mathbf{A}) \\ &= \lambda_{\max}(\mathbf{A}^2) \\ &\leq \|\mathbf{A}^2\|_1 \\ &\leq \|\mathbf{A}\|_1^2 \end{aligned} \quad (34)$$

where we have used the fact that the spectral radius of a matrix is bounded by its 1-norm. For monotonic convergence in the p-norm ($p=1,2$ or ∞) we need $\|\mathbf{F}\mathbf{e}_{k+1}\|_p \leq \|\mathbf{F}\mathbf{e}_k\|_p$ for all k .

$$\begin{aligned} \|\mathbf{F}\mathbf{e}_{k+1}\|_p &= \|\mathbf{A}\mathbf{F}\mathbf{e}_k\|_p \\ &\leq \|\mathbf{A}\|_p \|\mathbf{F}\mathbf{e}_k\|_p \\ &\leq \|\mathbf{A}\|_1 \|\mathbf{F}\mathbf{e}_k\|_p \end{aligned} \quad (35)$$

where we have used the relation $\|\mathbf{A}\|_2 \leq \|\mathbf{A}\|_\infty = \|\mathbf{A}\|_1$. Since the induced-1 norm equals the maximal column sum we have the sufficient condition for monotonic convergence

$$\|\mathbf{A}\|_1 = |a_0| + 2 \sum_{k=1}^{nq_e+nu} |a_k| < 1 \quad (36)$$

Hence we conclude that the error converges to the fixed point of the iteration $\mathbf{F}\mathbf{e}_\infty = \mathbf{0}$ if \mathbf{A} is stable. In addition if (36) is satisfied it does so in a monotonic decreasing fashion. As expected the convergence condition (26) is implied by the stronger monotonicity condition (36) as shown below.

$$\begin{aligned} |a_0 + 2 \sum_{k=1}^{nq_e+nu} a_k \cos(k\theta)| &\leq |a_0| + 2 \sum_{k=1}^{nq_e+nu} |a_k \cos(k\theta)| \\ &\leq |a_0| + 2 \sum_{k=1}^{nq_e+nu} |a_k| |\cos(k\theta)| \\ &\leq |a_0| + 2 \sum_{k=1}^{nq_e+nu} |a_k| \end{aligned} \quad (37)$$

Also the error vector $\mathbf{e}_\infty = \mathbf{0}$ only when the system is fully invertible i.e., $nu = 0$ and the resultant \mathbf{F} is a nonsingular matrix.

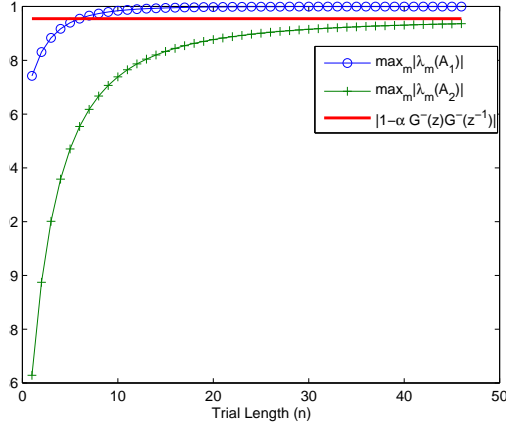


Fig. 1. Effect of “zero-padding” matrix on ILC stability

VII. NUMERICAL EXAMPLE

We shall illustrate the reason behind the introduction of the “zero-padding” matrix \mathbf{N} in section V with a simple example. Let us consider the following non-minimum phase plant [33]

$$y(t+1) = -0.2y(t) + 0.0125y(t-1) + u(t) - 1.1u(t-1)$$

From the definitions earlier (c.f. Section III-A) we have $G^-(z^{-1}) = (g_0 + g_1 z^{-1})$ where $g_0 = 1$ and $g_1 = -1.1$. The number of unstable zeros $nu = 1$. Now we use the proposed design methodology with $Q_u(z, z^{-1}) = Q_e(z, z^{-1}) = 1$ and $\alpha = 0.45$. Hence we have the noncausal filter representation for the transition matrix given by

$$\begin{aligned} A(z, z^{-1}) &= 1 - \alpha G^-(z)G^-(z^{-1}) \\ &= 1 - \alpha (g_0^2 + g_1^2 - g_0 g_1 z - g_0 g_1 z^{-1}) \end{aligned}$$

This choice satisfies the convergence condition (26)

$$\begin{aligned} |a_0 + 2a_1 \cos(\theta)| &< 1, \quad \forall \theta \in [0, \pi] \\ \Rightarrow |1 - \alpha(g_0^2 + g_1^2)| + 2|\alpha||g_0 g_1| &= 0.9955 < 1 \end{aligned} \quad (38)$$

since $a_0 = 1 - \alpha(g_0^2 + g_1^2)$ and $a_1 = -\alpha g_0 g_1$. We have the transition matrix without zero-padding given by $\mathbf{A}_1 = \mathbf{I} - \alpha(\mathbf{G}^-)^T \mathbf{G}^-$. Instead if we do the zero-padding we get the matrix $\mathbf{A}_2 = \mathbf{I} - \alpha \mathbf{N}^T (\mathbf{G}^-)^T \mathbf{G}^- \mathbf{N}$. To understand the value of the zero-padding, one can easily check for trial length $n = 3$,

$$\begin{aligned} \mathbf{A}_1 &= \begin{bmatrix} 0.0055 & 0.4950 & 0 \\ 0.4950 & 0.0055 & 0.4950 \\ 0 & 0.4950 & 0.5500 \end{bmatrix} \\ \mathbf{A}_2 &= \begin{bmatrix} 0.0055 & 0.4950 & 0 \\ 0.4950 & 0.0055 & 0.4950 \\ 0 & 0.4950 & 0.0055 \end{bmatrix} \end{aligned}$$

The “zero-padding” matrix \mathbf{N} ensures that the (n, n) entry of the transition matrix is 0.0055 thereby marking it a banded toeplitz matrix. Remarkable as it may seem, difference in this single entry results in $\lim_{n \rightarrow \infty} \max_m |\lambda_m(\mathbf{A}_1)| = 1$ whereas $\lim_{n \rightarrow \infty} \max_m |\lambda_m(\mathbf{A}_2)| = 0.9955$ i.e., the upper bound in (38) (see fig. 1). We conclude that the “zero-padding” matrix \mathbf{N} is critical in ensuring stability of the learning algorithm. An

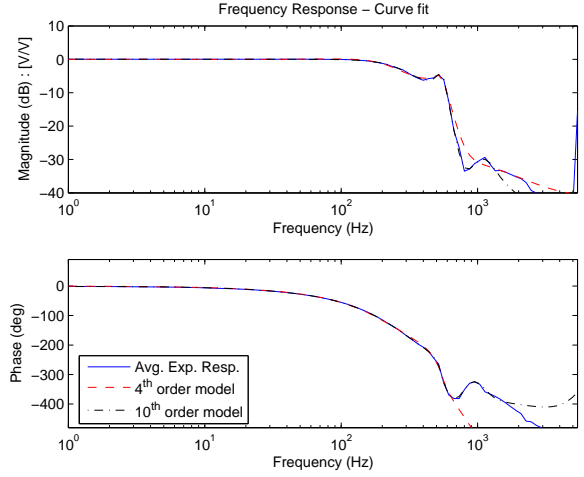


Fig. 2. PD Plant Experimental and Model Frequency Response

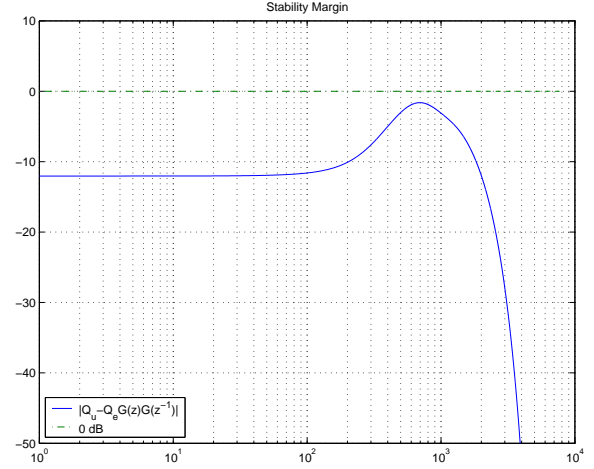


Fig. 3. Stability Condition $|Q_u(z, z^{-1}) - \alpha Q_e(z, z^{-1}) G^-(z^{-1}) G^-(z)|_{\infty} < 1$

injudicious choice of the learning matrices would result in instability despite the approximate stability condition (26) being met!

VIII. EXPERIMENTAL AND SIMULATION RESULTS

The plant to be controlled is a DC motor driven cutting tool with a stabilizing PD control. The system was identified in the discrete domain at a sampling frequency of $T=15\text{kHz}$. Figure 2 shows the PD plant and the curve fit model frequency response. We have a 4th order model which is used for control design and also a 10th order model which will be used in a later section. We will design a Iterative learning control to better the performance of the inner loop controller. As can be seen from the model frequency response (Fig. 2) this system has a low bandwidth since the phase drops quite sharply (100° by 200 Hz). The top plot of Fig. 4 shows the complex profile as a function of time. What is shown is the depth of cut that the tool tip has to achieve. The command profile has periodic content between $t = 1\text{s}$ and $t = 4\text{s}$ reflecting oval cross-section and flat regions between $t = 3.5\text{s}$ and $t = 3.8\text{s}$ reflecting round

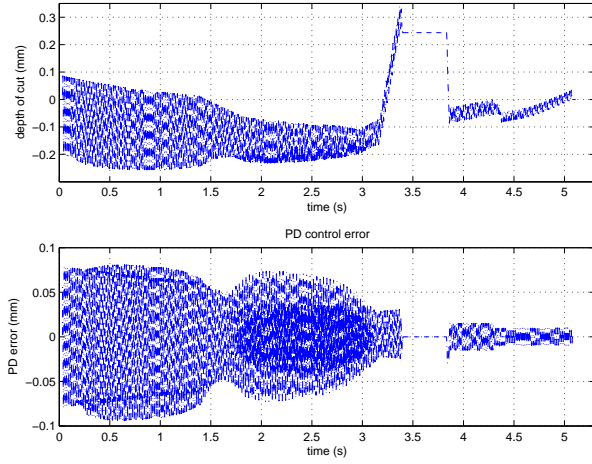


Fig. 4. Complex Tool Profile (top) with PD Tracking Error (bottom)

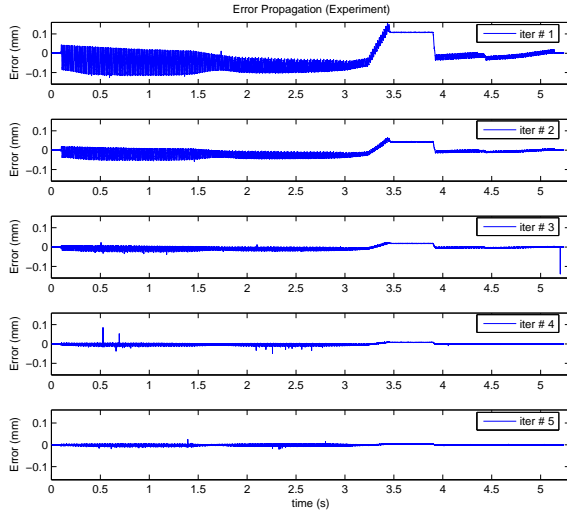


Fig. 5. Error Propagation for First Five Iterations (Experiment)

cross-section. The profile is complex in the sense that although it looks periodic, the magnitude and phase changes gradually making it hard to follow. Also phase compensation *a priori* based on the plant's frequency response (Fig. 2) is not possible due to the above reason. The bottom plot of Fig. 4 shows the error for the PD plant. The PD plant is not able to track this profile due to significant phase lag in the frequencies contained in the reference. We choose learning gain $\alpha = 0.75$ and zero phase low pass filters Q_u , Q_e of orders 16 and 32 respectively. Fig. 3 shows that the stability condition is met for this choice of learning matrices. We start the experiment with $\mathbf{u}'_0 = 0$, $\mathbf{e}_0 = \mathbf{r}$. Figure 5 shows the tracking error for the first five iterations. We see that the error decreases uniformly over the entire time axis. Fig. 6 shows the 10^{th} iteration tracking error achieved by the scheme. We also did a simulation with no model mismatch for comparison with the experimental results. The error is within an acceptable $\pm 10\mu\text{m}$ compared to $\pm 100\mu\text{m}$ achieved with the PD controlled plant (see bottom plot of Fig. 4).

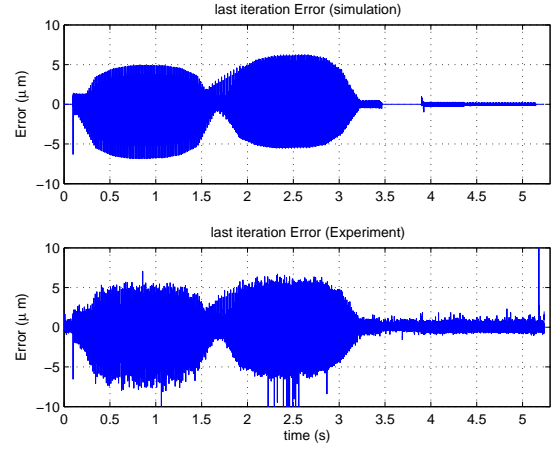


Fig. 6. Final Iteration Error Simulation (top) and Experiment (bottom)

A. Improvement over ZPETC Feedforward Control

Assuming the system was at rest $\mathbf{e}_0 = \mathbf{r}$ and $\mathbf{u}'_0 = 0$ we have

$$\mathbf{u}'_1 = \alpha(\mathbf{G}^-)^T \mathbf{r} \Rightarrow \mathbf{u}_1 = \alpha(\mathbf{G}^+)^{-1}(\mathbf{G}^-)^T \mathbf{r}$$

For the special choice of learning gain $\alpha = 1$ this is equivalent to the celebrated zero phase error tracking control (ZPETC) [16]. Hence we infer that the first iteration of the prototype ILC (see section III-A) corresponds to stable inversion based feedforward compensation. It is well known that feedforward control such as the ZPETC is sensitive to plant model uncertainties. But the prototype ILC has an inherent cycle-to-cycle feedback which makes it “robust” in a limited sense. As reported in [18] zero phase ILC can be seen as repeated application of the feedforward control leading to better performance despite the presence of model uncertainties. Also the learning gain α is seldom set to the aggressive value of 1. Instead we use a lower gain and multiple iterations to derive the same effect as feedforward control in a ideal plant model situation. To verify the above claim, we did a simulation using a 4^{th} order model of the plant for the control design. Then we applied ZPETC feedforward control to a more accurate 10^{th} order model (see fig. 2). Figure 7 shows that the feedforward error is within $\pm 30\mu\text{m}$ when there is a model mismatch (top) as against $\pm 5\mu\text{m}$ when there is perfect model match (bottom). Figure 8 shows that we can improve upon the model mismatch error using the proposed modified repetitive ILC and bring it within the acceptable $\pm 5\mu\text{m}$ region in as few as four iterations.

IX. CONCLUSIONS

We have introduced a noncausal ILC design based on zero-phase error compensation and filtering used in repetitive control design. The learning gains (matrices) are chosen carefully such that the resulting state transition matrix has a *Symmetric Banded Toeplitz* (SBT) structure. This special structure has been exploited to arrive at sufficient conditions for tracking error convergence in the frequency domain. Thus the ILC matrix design is translated to repetitive control loop shaping filter design in the frequency domain. For a special choice

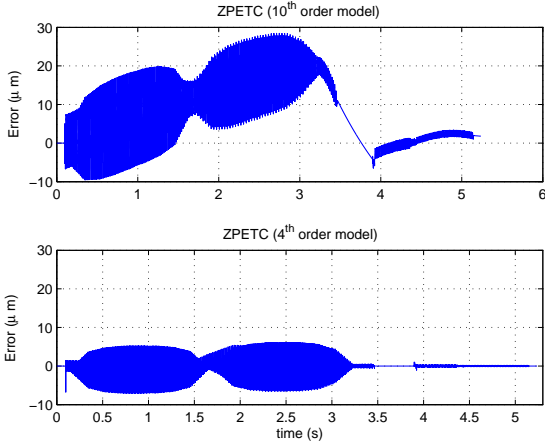


Fig. 7. 10th order model (top) 4th order model (bottom) ZPETC error

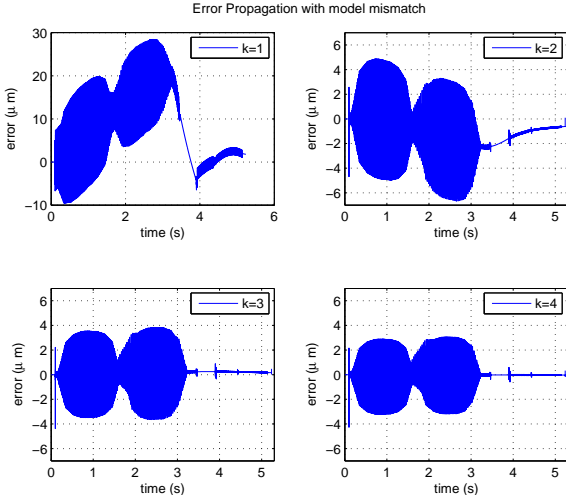


Fig. 8. Error Propagation with model mismatch (Simulation)

of learning matrix we have also derived a stronger monotonic convergence condition. Furthermore we have shown that under plant model mismatch the proposed scheme improves upon stable inversion based feedforward control. The proposed methodology has been successfully demonstrated on a fast tool servo system used in precision machining applications.

APPENDIX A

Theorem 1: The matrix $\mathbf{A} = \mathbf{Q}_u - \alpha \mathbf{N}^T (\mathbf{G}^-)^T \mathbf{Q}_e \mathbf{G}^- \mathbf{N}$ has a symmetric banded Toeplitz (SBT) structure

Proof: Recall the definitions

$$\begin{aligned} \mathbf{G}_{i,j}^- &= g_{i-j}, \quad 0 \leq i-j \leq nu \\ &= 0, \quad \text{otherwise} \\ (\mathbf{Q}_e)_{i,j} &= q_{|i-j|}^e, \quad |i-j| \leq nq \\ &= 0, \quad \text{otherwise} \\ \mathbf{N}_{i,j} &= 1, \quad 1 \leq j = i - nu \leq n \\ &= 0, \quad \text{otherwise} \end{aligned}$$

Hence we have the coefficients of $(\mathbf{Q}_e \mathbf{G}^-)_{n+2nu, n+2nu}$ given

by

$$\begin{aligned} b_{i,j} &= \sum_{k=\max(i-nq, j)}^{\min(i+nq^e, j+nu, n+2nu)} q_{|i-k|}^e g_{i-j}, \\ &\quad -nq^e \leq i-j \leq nq^e + nu \\ &= 0, \quad \text{otherwise} \end{aligned}$$

The coefficients of $(\mathbf{Q}_e \mathbf{G}^- \mathbf{N})_{n+2nu, n}$ are given by

$$\begin{aligned} c_{i,j} &= \\ b_{i,j+nu} &= \sum_{k=\max(i-nq^e, j+nu)}^{\min(i+nq^e, j+2nu)} q_{|i-k|}^e g_{k-j-nu}, \\ &\quad nu - nq^e \leq i-j \leq nq^e + 2nu \\ &= 0, \quad \text{otherwise} \end{aligned}$$

where we have used $j \leq n \Rightarrow j + 2nu \leq n + 2nu$.

The coefficients of $((\mathbf{G}^-)^T \mathbf{Q}_e \mathbf{G}^- \mathbf{N})_{n+2nu, n}$ are given by

$$\begin{aligned} d_{l,j} &= \sum_{i=1}^{n+2nu} (\mathbf{G}^-)_{i,l} c_{i,j} \\ &= \sum_{i=\max(l, j+nu-nq^e)}^{\min(l+nu, j+2nu+nq^e, n+2nu)} \{g_{i-l} \\ &\quad \sum_{k=\max(i-nq^e, j+nu)}^{\min(i+nq^e, j+2nu)} q_{|i-k|}^e g_{k-j-nu}\}, \\ &\quad -nq^e \leq l-j \leq nq^e + 2nu \\ &= 0, \quad \text{otherwise} \end{aligned}$$

The coefficients of $(\mathbf{N}^T (\mathbf{G}^-)^T \mathbf{Q}_e \mathbf{G}^- \mathbf{N})_{n,n}$ are given by

$$\begin{aligned} e_{l,j} &= \\ d_{l+nu, j} &= \sum_{i=\max(l+nu, j+nu-nq^e)}^{\min(l+2nu, j+2nu+nq^e)} \{g_{i-l-nu} \\ &\quad \sum_{k=\max(i-nq^e, j+nu)}^{\min(i+nq^e, j+2nu)} q_{|i-k|}^e g_{k-j-nu}\}, \\ &\quad |l-j| \leq nq^e + nu \\ &= 0, \quad \text{otherwise} \end{aligned}$$

where we have used $l \leq n \Rightarrow l + 2nu \leq n + 2nu$.

We do a change of variables $i' = i - nu$ and $k' = k - nu$ to get

$$\begin{aligned} e_{l,j} &= \sum_{i'=\max(l, j-nq^e)}^{\min(l+nu, j+nu+nq^e)} \{g_{i'-l} \\ &\quad \sum_{k'=\max(i'-nq^e, j)}^{\min(i'+nq^e, j+nu)} q_{|i'-k'|}^e g_{k'-j}\}, \\ &\quad |l-j| \leq nq^e + nu \\ &= 0, \quad \text{otherwise} \end{aligned}$$

We readily see that

$$\begin{aligned}
e_{l+1,j+1} &= \sum_{i'=\max(l+1,j+1-nq^e)}^{\min(l+1+nu,j+1+nu+nq^e)} \{g_{i'-l-1}\} \\
&\quad \sum_{k'=\max(i'-nq^e,j+1)}^{\min(i'+nq^e,j+1+nu)} q_{|i'-k'|}^e g_{k'-j-1} \} \\
&= \sum_{i=\max(l,j-nq^e)}^{\min(l+nu,j+nu+nq^e)} \{g_{i-l}\} \\
&\quad \sum_{k=\max(i-nq^e,j)}^{\min(i+nq^e,j+nu)} q_{|i-k|}^e g_{k-j} \} \\
&= e_{l,j}
\end{aligned}$$

by doing a change of variable $i = i' - 1$ and $k = k' - 1$. Hence $\mathbf{N}^T(\mathbf{G}^-)^T \mathbf{Q}_e \mathbf{G}^- \mathbf{N}$ (which is symmetric by definition) is a SBT matrix. Since \mathbf{Q}_u is by definition a SBT matrix, it follows that \mathbf{A} is also a SBT matrix. ■

APPENDIX B

Entries of the state transition matrix \mathbf{A} -

When $nq^u > nq^e + nu$ the coefficients of \mathbf{A} are given by

$$\begin{aligned}
f_{l,j} &= q_{|l-j|}^u - \alpha \sum_{i'=\max(l,j-nq^e)}^{\min(l+nu,j+nu+nq^e)} \{g_{i'-l}\} \\
&\quad \sum_{k'=\max(i'-nq^e,j)}^{\min(i'+nq^e,j+nu)} q_{|i'-k'|}^e g_{k'-j}, \\
&\quad |l-j| \leq nq^e + nu \\
&= q_{|l-j|}^u, \quad nq^e + nu < |l-j| \leq nq^u \\
&= 0, \quad \text{otherwise}
\end{aligned}$$

When $nq^u \leq nq^e + nu$ the coefficients of \mathbf{A} are given by

$$\begin{aligned}
f_{l,j} &= q_{|l-j|}^u - \alpha \sum_{i'=\max(l,j-nq^e)}^{\min(l+nu,j+nu+nq^e)} \{g_{i'-l}\} \\
&\quad \sum_{k'=\max(i'-nq^e,j)}^{\min(i'+nq^e,j+nu)} q_{|i'-k'|}^e g_{k'-j}, \\
&\quad |l-j| \leq nq^u \\
&= -\alpha \sum_{i'=\max(l,j-nq^e)}^{\min(l+nu,j+nu+nq^e)} \{g_{i'-l}\} \\
&\quad \sum_{k'=\max(i'-nq^e,j)}^{\min(i'+nq^e,j+nu)} q_{|i'-k'|}^e g_{k'-j}, \\
&\quad nq^u < |l-j| \leq nq^e + nu \\
&= 0, \quad \text{otherwise}
\end{aligned}$$

We get the entries in section V from the first row of the matrix

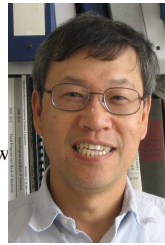
$$a_k = f_{1,k+1}, \quad 0 \leq k \leq r$$

where $r = \max(nq^u, nq^e + nu)$

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